Assessing the Risk and Return of Financial Trading Systems –
a Large Deviation Approach

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Trading System

- Defined by:
  - Entry condition(s)
  - Exit condition(s)
  - Position sizing
- Implemented in an Automated Trading System (ATS) or executed by a trader
Performance of a trading system

- **Performance metric**: return (P&L), Sharpe ratio, ...
- **Reference period** – e.g.: day, week, ...

<table>
<thead>
<tr>
<th>P&amp;L interval</th>
<th>Probability</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-3, -2]$</td>
<td>1/25</td>
<td>2.5</td>
</tr>
<tr>
<td>$(-2, -1]$</td>
<td>2/25</td>
<td>1.5</td>
</tr>
<tr>
<td>$(-1, -0]$</td>
<td>3/25</td>
<td>0.5</td>
</tr>
<tr>
<td>$(0, 1]$</td>
<td>12/25</td>
<td>0.5</td>
</tr>
<tr>
<td>$(1, 2]$</td>
<td>4/25</td>
<td>1.5</td>
</tr>
<tr>
<td>$(2, 3]$</td>
<td>2/25</td>
<td>2.5</td>
</tr>
<tr>
<td>$(3, 4]$</td>
<td>1/25</td>
<td>3.5</td>
</tr>
</tbody>
</table>

Distribution of the performance metric

Obtaining the distribution of the performance metric

1. Prior use of the TS
2. Back-testing on historical data, but:
   - **Does not account for slippage and delays**
   - **Data-mining bias** if a large number of systems are tested
   - Performance must be adjusted accordingly!
Notations and Assumptions

$X_i$: performance at period $i$

$X_1, X_2, \ldots, X_n$: 1) are mutually independent and identically distributed
2) obey a distribution law that does not change over time

How to assess the risks?

- We want to estimate $p = P\left[\sum_{i}^{n} X_i < x\right]$

1. Monte-Carlo simulation
2. Analytic approaches:
   1. Markov’s, Tchebychev’s, Chernoff’s upper bounds
   2. Large deviation
Monte-Carlo simulation

- Generate $n$ random trading sequences and compute an estimate of the probability
- CLT tells us that the estimate will convergence to $p$ but slowly and
  
  \[ \text{percentage error} = \frac{\sqrt{p(1-p)}}{\sqrt{np}} \cdot 100 \]

- Error bound of 1% with $p=10^{-5}$ requires $n=10^9$
- Problem: random number generators are not perfect..

Analytic approaches

- Weak law of large numbers:
  \[ \lim_{n \to \infty} P \left( \left| \frac{\sum_{i=1}^{n} X_i}{n} - E[X] \right| < \varepsilon \right) \forall \varepsilon > 0 \]
  
  But the rate of convergence is unknown..

- Elements of solution:
  1. Markov’s inequality: \[ \forall \alpha \quad P (X \geq \alpha E[X]) \leq \frac{1}{\alpha} \]
  2. Tchebychev’s inequality: \[ P \left( |X - E[X]| \geq k\sigma[X] \right) \leq \frac{1}{k^2} \]
    
    Not tight enough for real-world applications
Large deviation: main result

\[ M_n = \frac{1}{n} \sum_{i=1}^{n} X_i \] : mean performance over \( n \) periods

- **Cramer's theorem**: if \( X_n \) i.i.d. r.v.

\[ P(M_n \in G) \asymp e^{-n \inf_{x \in G} I(x)} \]

with \( I(x) \) the rate function

\[ I(x) = \sup_{\tau > 0} \left( \tau x - \log E(e^{\tau X}) \right) = \sup_{\tau > 0} \left( \tau x - \log \sum_{k=-\infty}^{+\infty} p_k e^{k\tau} \right) \]

Technical contribution

- Can deal with distributions given as frequency histogram (no closed-form)
  - \( I(x) \) is the sup. of affine functions and thus convex
  - Computing the point where first derivative equal zero is thus enough
  - Can be done with standard numerical methods
Risk over a given time interval

![Graph showing risk over time intervals]

- $P[M_n < 0] \leq 0.001$
- $P[M_n < -0.5K] \leq 0.04$
- $P[M_n < -1K] \leq 0.001$

Quantifying the uncertainty

- The uncertainty of trading system $Sp$ to achieve a performance $x$ over $n$ time periods is
  \[ U(x, n) = P[M_n \leq x/n] \leq e^{-n \inf_y \leq x/n I(y)} \]

- $Sp$ is with performance objective $x$ over $n$ time periods is less uncertain than $Sp'$ with return objective $x'$ over $n'$ time periods if
  \[ U(x, n) \leq U'(x', n') \]
Detecting changing market conditions

- Idea: if a TS performs way below what was foreseeable, it suggests that market conditions have changed

E.g., if the current performance level had a probability less than $10^{-6}$

Portfolio of Trading Systems

- **Assumption:** TS are independent

- Comes to evaluate: $P\left[ \frac{1}{n} \sum_{i=1}^{n} (X_1^i + X_2^i + ... + X_m^i) < x \right]$  

- Sum of 2 id. r.v. = convolution, computed using Fast Fourier Transform:

  \[ f \ast g = FFT^{-1}(FFT(x) \cdot FFT(y)) \]
Conclusion

- LD is better suited than simulation for rare events ($<10^{-4}$)
- LD can serve to validate simulation results
- LD helps to detect changing market conditions
- Our approach is practical:
  - No need for closed-form distributions
  - Easily implementable
  - Work for portfolio of TS
  - Can be embedded in a broader analysis

Extensions

- There are ways to address the cases:
  - There are serial dependencies in the trade outcomes
  - The market conditions are changing over time
    ➔ p.d.f. non-homogeneous in time